(2.34) Partial Derivatives of Higher Orders.

The partial derivatives f_x and f_y of a function f of two variables x and y, being functions of x and y, may possess derivatives. In such cases, the **second order partial derivatives** are defined as below.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx} = f_{x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy} = f_{y^2}.$$

Thus, there are four second order partial derivatives for a function $z = f(x, y)$. The partial derivatives f_{xy} and f_{yx} are called **mixed second partials** and are **not equal** in general. Partial derivatives of orders higher than two can be defined in a similar manner.

Example 33. Find the first order partial derivatives of

$$z = \ln\left(\frac{x^2 + y^2}{x + y}\right)$$

Solution.

$$z = \ln(x^2 + y^2) - \ln(x + y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2} \cdot 2x - \frac{1}{x + y}$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x^2 + y^2)(x + y)} = \frac{x^2 + 2xy - y^2}{(x^2 + y^2)(x + y)}$$

Similarly, (or by symmetry)

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x^2 + y^2)(x + y)}.$$

Example 34. Let $z = \arcsin\left(\frac{x}{y}\right)$. Verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Solution.

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} = \frac{1}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{-x}{y^2} = \frac{-x}{y\sqrt{y^2 - x^2}}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{-1}{2}(y^2-x^2)^{-3/2}\cdot 2y = \frac{-y}{(y^2-x^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{-1}{y\sqrt{y^2-x^2}} - \frac{x}{y}\left[\frac{x}{(y^2-x^2)^{3/2}}\right]$$

$$= \frac{-y^2+x^2-x^2}{y(y^2-x^2)^{3/2}} = \frac{-y}{(y^2-x^2)^{3/2}}$$

From (1) and (2) it is clear that $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$.

Example 35. If $f(x, y) = e^x \sin y + e^y \cos x$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

(1) is called ***Laplace's equation.**

Solution. $\dfrac{\partial f}{\partial x} = e^x \sin y - e^y \sin x$

$$\frac{\partial^2 f}{\partial x^2} = e^x \sin y - e^y \cos x$$

$$\frac{\partial f}{\partial y} = e^x \cos y + e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \sin y + e^y \cos x$$

Adding (2) and (3), we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x \sin y - e^y \cos x - e^x \sin y + e^y \cos x = 0$$

Example 36. [illegible]

Find all the four second order partial derivatives (Problems 23 – 26):

23. e^{x-y}

24. $\dfrac{x+y}{x-y}$

25. e^{xy}

26. $\tan(\arctan x + \arctan y)$

In Problems 27 – 32 verify that $f_{xy} = f_{yx}$

27. $f(x, y) = e^{xy}\cos(bx + c)$

28. $f(x, y) = \ln(e^x + e^y)$

29. $f(x, y) = \ln\left(\dfrac{x^2 + y^2}{xy}\right)$

30. $f(x, y) = x^y + y^x$

31. $f(x, y) = x \sin xy + y \cos xy$

32. $f(x, y) = \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$

Show that each of the following functions satisfies Laplace's equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. (Problem 33 – 36):

33. $f(x, y) = \sin x \sinh y$

34. $f(x, y) = e^{-x}\cos y$

35. $f(x, y) = \ln\sqrt{x^2 + y^2}$

36. $f(x, y) = \arctan\left(\dfrac{2xy}{x^2 - y^2}\right)$

37. If $f(x, y) = x^2 \arctan\left(\dfrac{y}{x}\right) - y^2 \arctan\left(\dfrac{x}{y}\right)$, show that

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$

38. If $f(x, y) = \dfrac{x^2 + y^2}{x + y}$, prove that

$$(f_x - f_y)^2 = 4(1 - f_x - f_y)$$

39. Show that the function $f(x, y) = \sin xy$ satisfies the differential equation

$$x^2 f_{xx} - y^2 f_{yy} = 0.$$

40. Let $f(x, y) = \begin{cases} x^2 \arctan\left(\dfrac{y}{x}\right) - y^2 \arctan\left(\dfrac{x}{y}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

41. (i) Let $f(x, y, z) = x^3 + 3yz + \sin xyz$. Prove that $f_{xyz} = f_{zxy}$.

(ii) If $f(x, y, z, w) = \dfrac{xy}{z + w}$, show that $f_{xyzw} = \dfrac{2}{(x + w)^3}$.

In Problems 42 – 45, find $\dfrac{dy}{dx}$ by using partial derivatives:

42. $y^2 + x^2 y + ax^4 = 0$

43. $3x^2 - y^2 + x^3 = 0$

44. $x^2 + xy + y^2 + ax + by = 0$

45. $x^3 + x^2 + xy^2 + \sin y = 0.$